USA TST 2007

Problem:Triangle ABC incribed in cirlcle ω.The tangent lines to ω at B and C meet at T.Point S lies on BC such that $AS \perp AT$.Points B_1 and C_1 lies on ray ST (with C_1 between S and T) such that $B_1T = BT = C_1T$.Prove that $\triangle ABC \sim \triangle AB_1C_1$.

Proof:

It's easy to notice B, B_1, C_1, C are concyclic.From now we'll call this circle ω_2.Let B_1B again meet ω at F.Let FC again meet circle ω_2 at C_2.Let the center of ω is O.Notice that O, B, T, C are concyclic.So $\angle BFC + \angle \mathrm{BC}_2C = \frac{1}{2}(\angle BOC + \angle BTC) = \frac{\pi}{2}$.Hence $\angle C_2BF = \angle C_2BB_1 = \frac{\pi}{2}$.So B_1C_2 is the dimeter of ω_2.Hence $C_1 = C_2$.As $BF \perp BC_1$ and $CF \perp CB_1$, it follows that F, B, R, C concyclic.($R = BC_1 \cap CB_1$)

Let TR again meet ω at A'.From power of point T wrt ω, $BT^2 = A'T.RT = TR.TA'$.So R and A' are inversive points wrt ω_2.Now in cyclic quad BB_1C_1C, Brochard's theorem and Li Hire's theorem give together,

$$R \in \text{polar of } S \Leftrightarrow S \in \text{polar of } R \Rightarrow \angle TA'S = \frac{\pi}{2}$$

Also remember $\angle TAS = \frac{\pi}{2}$.So A, A', T, S concyclic.Hence if $A \neq A'$ then A' must be the second intersection point of ω and the circumcircle of $\triangle ATS$ (We'll call this circle ω_3).Let the mid point of BC is D.So it follows that $\angle TDS = \frac{\pi}{2}$.So D also lies on circle ω_3.Notice that as R lies on arc BRC of ω,A' must lie on arc BAC.(As arc BRC lies inside circle ω_2.)Notice that T lies inside of $\angle BAC$.As ω_3 passes through points A, D, T and ω passes through points B, C so it follows that the circles must intersect again inside $\angle BAC$.(That is if you extend ray AB and AC, then they cut out a portion from the whole plane.The second intersection point will lie on this portion).It implies A', the second intersection point of ω and ω_3, must lie on arc BRC which is impossible.So $A = A'$. So $\angle CC_1S = \frac{\pi}{2} + \angle C_1BC = \frac{\pi}{2} + \angle RBC = \frac{\pi}{2} + \angle RAC = \frac{\pi}{2} + \frac{\pi}{2} - \angle CAS = \pi - \angle CAS$

Therefore, A, C, C_1, S concyclic.Similarly, A, B, B_1, S concyclic.Hence $\angle ABS = \angle AB_1S$ and $\angle ACS = \angle AC_1S$.Now the result easily follows.