Pole-Polar: *Key Facts*

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In this article, we shall provide key facts of Pole-Polar transformation, however without proof. Interested readers may find the proofs in various sources, referred sources are certainly good places to start. The main goal of this article is to provide a tool kit for problem solver.

1 Basic Facts

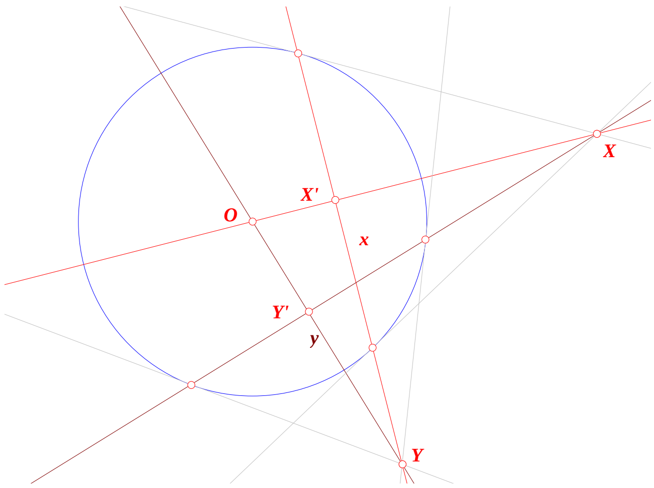
Theorem 1.1 (La Hire's Theorem)**.** *Let x and y be the polars of X and Y, respectively.*

Then X is on line $y \iff Y$ is on line x.

Proof. Let X', Y' be the images of X, Y for the inversion with respect to C. Then $OX.OX' = r^2 = OY.OY' \implies X.X', Y, Y'$ are concyclic. Now,

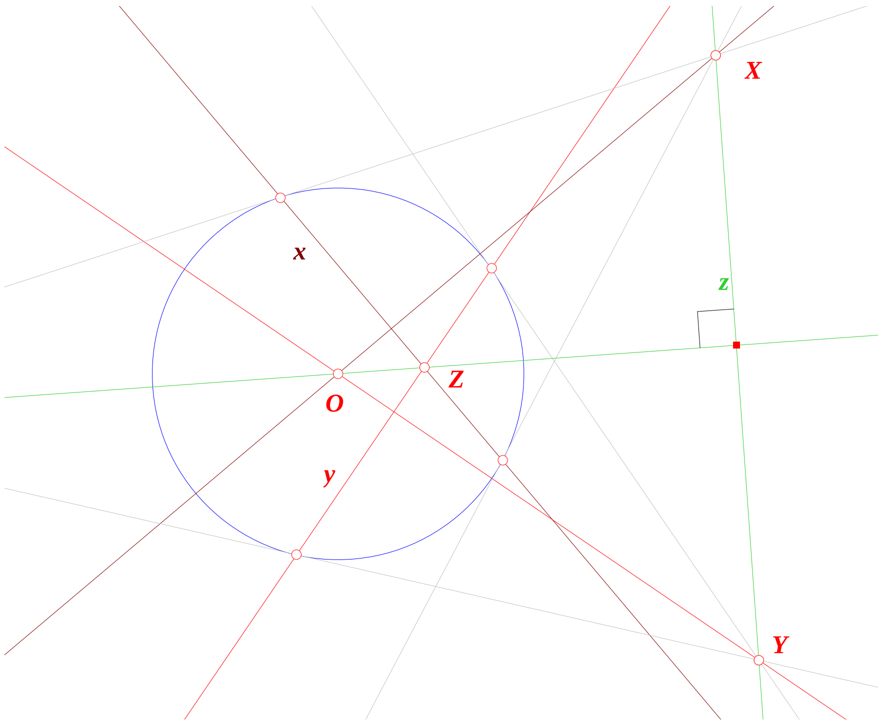
$$\begin{aligned} X \in y &\iff \angle XY'Y = 90^o \\ &\iff \angle XX'Y = 90^o \\ &\iff Y \in x \end{aligned}$$

□



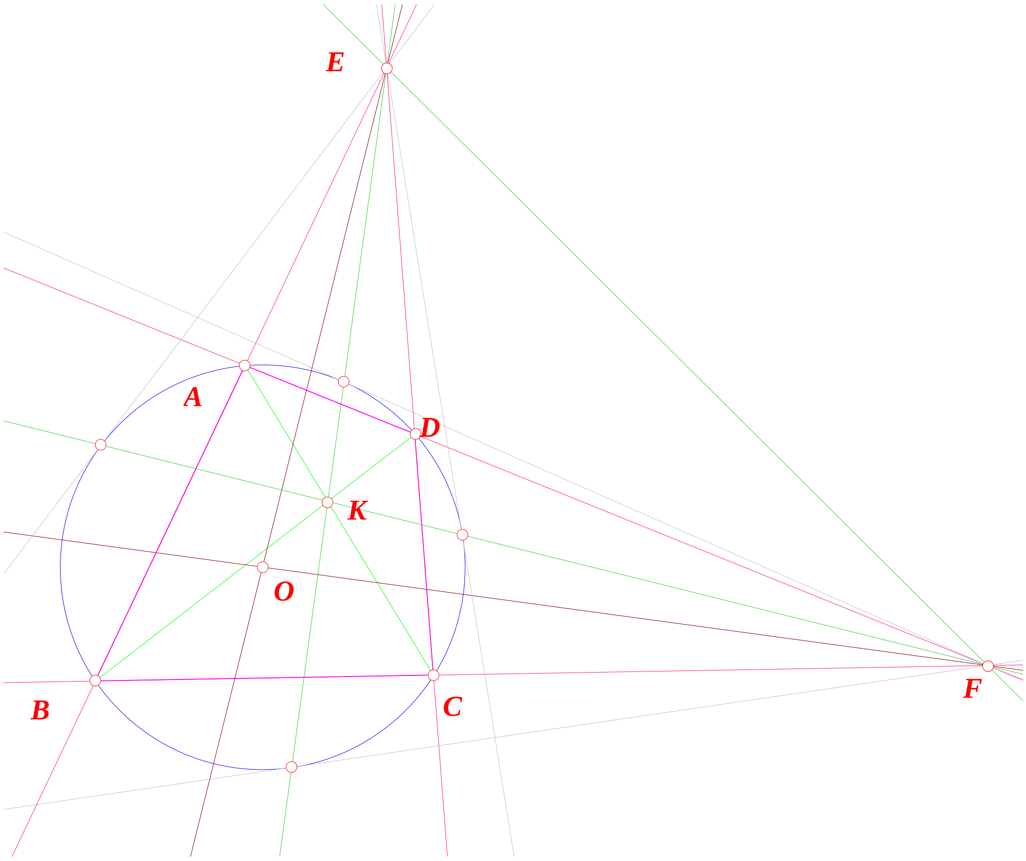
Theorem 1.2. *Let x, y, z be the polars of distinct points X, Y, Z respectively. Then $Z = x \cap y \iff z = XY$*

Proof. By La Hire's theorem, Z on $x \cap y \iff X$ on z and Y on $z \iff z = XY$. □



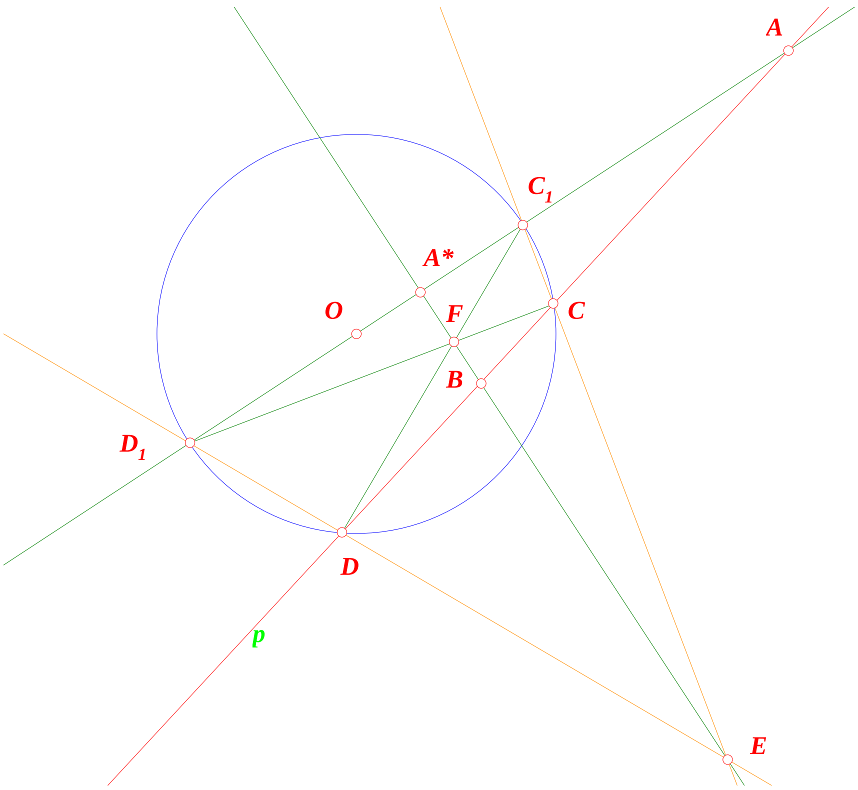
Theorem 1.3. *Let $ABCD$ be a cyclic quadrilateral and (O) be its circumcircle. Denote points $E = AB \cap CD$, $F = AD \cap BC$ and $K = AC \cap BD$. Then prove that:*

- *EF is the polar of K w.r.t. (O)*
- *EK is the polar of F w.r.t. (O)*
- *FK is the polar of E w.r.t. (O)*



Theorem 1.4 (Brokard)**.** *O is the orthocentre of triangle EFK.*

Theorem 1.5. *If $EA^* \perp AO$, EA^* is the polar of the pole A, iff $(C_1D_1A^*A) = -1$.*



Theorem 1.6. *The polars of three points are concurrent if and only if they are collinear*

Proof. ($\Leftarrow$) let X, Y, Z be three points on line d and let D be the pole of line d wrt circle w,now the polar of D passes through X so the polar of X passes through D,in the same way we conclude that the polars of Y, Z pass through D,so the polars of three collinear points X, Y, Z are concurrent.
($\Rightarrow$) let x, y, z be three concurrent lines at D,and let d be the polar of D wrt circle w and let X, Y, Z be the poles of x, y, z respectively,now x passes through D so the polar of point D passes through the pole of line x i.e. d passes through X in the same way we conclude that d passes through Y, Z so X, Y, Z lie on line d which means that the poles of three concurrent lines are collinear. □

1.1 A few words on ***Harmonic Conjugates*** and ***Cross Ratio.***

Theorem 1.7. ***Inversion*** *and* ***Projectivity*** *preserves cross ratio and hence harmonic conjugates.*

Theorem 1.8. *Assume that the points A, B, C, D_1, and D_2 are either collinear or concyclic. If $\mathcal{R}(A, B; C, D_1) = \mathcal{R}(A, B; C, D_2)$, then $D_1 = D2$. In other words, a projectivity with three fixed points is the identity.*

Theorem 1.9. *If the points A, B, C, D are mutually disjoint and $\mathcal{R}(A, B; C, D) = \mathcal{R}(B, A; C, D)$ then $\mathcal{H}(A, B; C, D)$.*

2 Examples

Problem 1 Let $ABCD$ is a quadrilateral and $ABCD$ inscribed in a circle (O). Let AC meet BD at point I. Let d is a line through point I and $d \perp OI$. Let d meet AD and BC at E and F respectively. Prove that I is midpoint of EF.

Solution 1 Suppose that

$$AD \cap BC = P \qquad BA \cap CD = Q \qquad PI \cap CD = Z$$

It is well-known fact that both points P, Q lies on the polar of I w.r.t to circumcircle of $ABCD$.Therefore,PQ is the polar of I. It means that $OI \perp PQ$,simultaneously $OI \perp EF$,hence $PQ \| EF$.

As we know division (Q, Z, C, D) is harmonic. Now consider a pencil $P(X_\infty, I, F, E)$ and its intersection with EF.

Finally,$(X_\infty, I, F, E) = -1$

Hence $EI = IF$ and we are done.

Solution 2 If EF meets the circle at K, L , then I is the midpoint of KL. By the butterfly theorem (with chord AC, BD through the midpoint I of KL we have then : $IE = IF$

Problem 2 [IMO shortlist] Let $ABCD$ be a bicentric quadrilateral, with (O) be its circumcircle, (I) be its incircle and H the intersection of AC, BD. Prove that O, I, H are collinear.

Solution Denote $F \in CB \cap AD$,$G \in AB \cap DC$.We know that H lies on the polars of F, G w.r.t (I) and (O),hence FG is a polar of H w.r.t (I),(O).Now it is obvious,because $IH \perp FG$ and $OH \perp FG$.

Problem 3 Let $ABCD$ be a quadrilateral, which has an inscribed circle (O). Let H be the orthogonal projection of O onto BD. Prove that $\angle AHB = \angle CHB$.

Solution Proof: Denote by A_1, B_1, C_1, D_1 the points of tangency of the incircle with the sides AB, BC, CD, DA respectively.
Let X be the intersection point of A_1B_1 and C_1D_1 and Y be the intersection point of B_1C_1 and A_1D_1.Thus BD is the polar of X w.r.t (O) and AC is a polar of Y w.r.t (O).It is well-known that X also lies on the polar of Y w.r.t (O).Hence A, C, X are collinear.
Denote $Z \in XA_1 \cap YI$,where $I \in A_1C_1 \cap B_1D_1$.We also know that I lies on the polar of Y w.r.t to (O).Hence A, I, C, X are collinear.Since $(A_1, Z, B_1, X) = -1 \Rightarrow (A, I, C, X) = -1$.(Consider the pencil $B(A_1, Z, B_1, X)$).
Now it sufficiently to show that O, H, X are collinear,which is obvious,since BD is a polar of X w.r.t (O).

Problem 4 Let w be the incircle of triangle $\triangle ABC$ and let w touch the side BC at D and let H be a point on BC such that $AH \perp BC$ and name the midpoint of AH as M. Now let DM intersects w at X, prove that the circumcircle of $\triangle BXC$ is tangent to w.

Solution Let E, F be the tangency points of ω with the sides CA, AB, respectively. Call $D' = EF \cap BC$, and Y the mid-point of the segment DD'. We have AD is the polar line of D' w.r.t. ω, so $ID' \perp AD$, where I is the center of ω. It implies $\triangle ADH \sim \triangle D'ID$, $\Rightarrow \triangle MDH \sim \triangle YID$. It means $IY \perp DM$, i.e. DM is the polar line of Y relative to ω. Hence, YX is tangent to ω (1).
On the other hand, the division $(BDCD')$ is harmonic, so $YB \cdot YC = YD^2 = YX^2$. It follows YX is tangent to the circumcircle of triangle BXC (2).
Combining (1) and (2) we get the result.

Remark. If we define N, P as the mid-points of the B - altitude and C - altitude of triangle ABC, then we also have DM, EN and FP are concurrent.

Problem 5 let the incircle of triangle $\triangle ABC$ touch the sides BC, CA, AB at D, E, F respectively,let EF intersect BC at A',define B', C' in the same way,prove that A', B', C' are collinear.

Solution Polar of A' is AD (Because $(AB, AC, AD, AA') = -1$), polar of B' is BE, polar of C' is CF. Because AD, BE, CF are concurrent then A', B', C' are collinear.

Problem 6]Let P be a point outside circle w and let two lines passing through P intersect w at A, B and C, D respectively (so P, A, B and P, C, D are collinear and also B lies between P, A and C lies between P, D),now let AD, BC intersect at Q and let AC, BD intersect at S,prove that:
1)$OS \perp PQ$
2)$QS \perp OP$

Solution 1.We know that S lies on the polar of P and on the polar Q w.r.t (O),hence S is a pole of PQ w.r.t to (O),thus $OS \perp QP$.

2.Points Q and S lie on the polar of P w.r.t (O),hence $QS \perp OP$.

More examples can be found in [1],[2],[3].

3 References

1. Pole and Polar, *Kin Y. Li*
2. Projective Geometry, *Milivoje Lukic*
3. Examples of using harmonic divisions and polarity,
   `http://www.mathlinks.ro/viewtopic.php?t=168866`
4. An Introduction to Projective Geometry, *Bobby Poon*

* This document is prepared using LaTeX.

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