Projective Geometry - Part 2

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Review

- Four collinear points A, B, C, D form a *harmonic bundle* $(A, C; B, D)$ when $\frac{\overrightarrow{CA}}{\overrightarrow{CB}} : \frac{\overrightarrow{DA}}{\overrightarrow{DB}} = -1$.
- A *pencil* $P(A, B, C, D)$ is the set of four lines PA, PB, PC, PD. It is harmonic iff $(A, B; C, D)$ is harmonic. Intersecting a harmonic pencil with any line produces a harmonic bundle.
- In $\triangle ABC$, points D, E, F are on sides BC, CA, AB. Let FE intersect BC at G. Then $(B, C; D, G)$ is harmonic iff AD, BE, CF are concurrent.
- A point P is outside or on a circle ω. Let PC, PD be tangents to ω, and l be a line through P intersecting ω at A, B (so that P, A, B are collinear in this order). Let AB intersect CD at Q. Then $ACBD$ is a harmonic quadrilateral (i.e. $\frac{AC}{CB} = \frac{AD}{DB}$) and $(P, Q; A, B)$ is harmonic.
- Points A, C, B, D lie on a line in this order, and M is the midpoint of CD. Then $(A, B; C, D)$ is harmonic iff $AC \cdot AD = AB \cdot AM$. Furthermore, if $(A, B; C, D)$ is harmonic then $MD^2 = MA \cdot MB$.
- Points A, C, B, D lie on a line in this order. P is a point not on on this line. Then any two of the following conditions imply the third:
  1. $(A, B; C, D)$ is harmonic.
  2. PB is the angle bisector of $\angle CPD$.
  3. $AP \perp PB$.
- Given a circle ω with center O and radius r and any point $A \neq O$. Let A' be the point on ray OA such that $OA \cdot OA' = r^2$. The line l through A' perpendicular to OA is called the *polar* of A with respect to ω. A is called the *pole* of l with respect to ω.
- Consider a circle ω and a point P outside it. Let PC and PD be the tangents from P to ω. Then ST is the polar of P with respect to ω.
- **La Hire's Theorem:** A point X lies on the polar of a point Y with respect to a circle ω. Then Y lies on the polar of X with respect to ω.
- **Brokard's Theorem:** The points A, B, C, D lie in this order on a circle ω with center O. AC and BD intersect at P, AB and DC intersect at Q, AD and BC intersect at R. Then O is the orthocenter of $\triangle PQR$. Furthermore, QR is the polar of P, PQ is the polar of R, and PR is the polar of Q with respect to ω.
- M is the midpoint of a line segment AB. Let P_∞ be a point at infinity on line AB. Then $(M, P_\infty; A, B)$ is harmonic.

Heavy Machinery

- For a point P and a circle ω with center O, radius r, define the power of a point P with respect to ω by $d(P, \omega) = PO^2 - r^2$. For two circles ω_1, ω_2 there exists a unique line l, called the **radical axis**, such that the powers of any point on this line with respect to ω_1, ω_2 are equal. In particular, if $\omega_1 \cap \omega_2 = \{P, Q\}$ then line PQ is the radical axis of ω_1, ω_2.
  Radical Axis Theorem: Given three circles $\omega_1, \omega_2, \omega_3$, let l, m, n be the radical axes of $\omega_1, \omega_2; \omega_1, \omega_3; \omega_2, \omega_3$ respectively. Then l, m, n are concurrent at a point called the **radical centre** of the three circles.

- **Pascal's Theorem**: Given a hexagon $ABCDEF$ inscribed in a circle, let $P = AB \cap ED, Q = BC \cap EF, R = CD \cap AF$. Then P, Q, R are collinear. (An easy way to remember - the three points of intersection of pairs of opposite sides are collinear).
  Note: Points A, B, C, D, E, F do not have to lie on the circle in this order.
  Note: It is sometimes useful to use degenerate versions of Pascal's Theorem. For example if $C \equiv D$ then line CD becomes the tangent to the circle at C.

- **Brianchon's Theorem**: Given a hexagon $ABCDEF$ circumscribed about a circle, the three diagonals joining pairs of opposite points are concurrent, i.e. AD, BE, CF are concurrent.
  Note: It is sometimes useful to use degenerate versions of Brianchon's Theorem. For example if $ABCD$ is a quadrilateral circumscribed about a circle tangent to BC, AD at P, Q then PQ, AC, BD are concurrent.

- **Desargues' Theorem**: Given two triangles $A_1B_1C_1$ and $A_2B_2C_2$ we say that they are perspective with respect to a point when A_1A_2, B_1B_2, C_1C_2 are concurrent. We say that they are perspective with respect to a line when $A_1B_1 \cap A_2B_2, A_1C_1 \cap A_2C_2, C_1B_1 \cap C_2B_2$ are collinear. Then two triangles are perspective with respect to a point iff they are perspective with respect to a line.

- **Sawayama-Thebault's Theorem**: A point D is on side BC of $\triangle ABC$. A circle ω_1 with centre O_1 is tangent to AD, BD and Γ, the circumcircle of $\triangle ABC$. A circle ω_2 with centre O_2 is tangent to AD, DC and Γ. Let I be the incentre of $\triangle ABC$. Then O_1, I, O_2 are collinear. It is unlikely that a problem using this theorem will come up on IMO, however it is a nice result and is a good exercise to prove. See problem 5.

Homothety

Looking at geometric configurations in terms of various geometric transformations often offers great insight in the problem. You should be able to recognize configurations where transformations can be applied, such as homothety, reflections, spiral similarities, and projective transformations. Today we will be focusing on homothety. The powerful thing about homothety is that it preserves angles and tangency.

Consider two circles ω_1, ω_2 with centres O_1, O_2. There are two unique points P, Q, such that a homothety with centre P and positive coefficient carries ω_1 to ω_2, and a a homothety with centre Q and negative coefficient carries ω_1 to ω_2. P is called the *exsimilicentre*, and Q is called the *insimilicentre* of ω_1, ω_2. Some useful facts:

1. P is the intersection of external tangents to ω_1, ω_2. Q is the intersection of internal tangents to ω_1, ω_2.

2. Let ω_1, ω_2 intersect at S, R; PA_1, PA_2 are tangents to ω_1, ω_2 so that A_1, A_2 are on the same side of O_1O_2 as S. Then PR is tangent to the circumcircle of $\triangle A_1RA_2$.

3. $(P, R; O_1, O_2)$ is harmonic.

4. **Monge's Theorem**: Given three circles $\omega_1, \omega_2, \omega_3$. Then the exsimilicentres of ω_1 and ω_2, of ω_1 and ω_3, and of ω_2 and ω_3 are collinear.
   Proof: Let O_1, O_2, O_3 be the centres of the circles. Let K_1 be the intersection of the common tangents of ω_1, ω_2 and ω_1, ω_3. Define K_2, K_3 similarly. Then K_iA_i is the angle bisector of $\angle K_i$ in $\triangle K_1K_2K_3$. Hence K_1A_1, K_2A_2, K_3A_3 are concurrent. The result follows by Desargues' theorem.
   A proof without using Desargues' theorem: let X_3 be the exsimilicentre of ω_1, ω_2; define X_1, X_2 similarly. Apply Menelaus Theorem to $\triangle X_1X_2O_3$.

5. **Monge-d'Alembert Theorem**: Given three circles $\omega_1, \omega_2, \omega_3$. Then the exsimilicentre of ω_1, ω_2, the insimilicentre of ω_1, ω_3 and the insimilicentre of ω_2, ω_3 are collinear.
   Proof: Let O_1, O_2, O_3 be the centres of the circles; X_3 be the exsimilicentre of ω_1, ω_2; define X_1, X_2 similarly. Apply Menelaus Theorem to $\triangle O_1O_2O_3$.

Problems

Some of these problems are lemmas from Yufei Zhao's handout on Lemmas in Eucledian Geometry. The lemmas cannot be quoted on a math contest, so make sure to know their proofs!

1. The incircle ω of $\triangle ABC$ has centre I and touches BC at D. DE is the diameter of ω. If AE intersects BC at F, prove that $BD = FC$.

2. The incircle of $\triangle ABC$ touches BC at E. AD is the altitude in $\triangle ABC$; M is the midpoint of AD. Let I_a be the centre of the excircle opposite to A of $\triangle ABC$. Prove that M, E, I_a are collinear.

3. A circle ω is internally tangent to a circle Γ at P. A and B are points on Γ such that AB is tangent to ω at K. Show that PK bisects the arc AB not containing point P.

4. Let Γ be the circumcircle of $\triangle ABC$ and D an arbitrary point on side BC. The circle ω is tangent to AD, DC, Γ at F, E, K respectively. Prove that the incentre I of $\triangle ABC$ lies on EF.

5. Prove the Sawayama-Thebault's Theorem.

6. Γ is the circumcircle of $\triangle ABC$. The incircle ω is tangent to BC, CA, AB at D, E, F respectively. A circle ω_A is tangent to BC at D and to Γ at A', so that A' and A are on different sides of BC. Define B', C' similarly. Prove that DA', EB', FC' are concurrent.

7. (Romania TST 2004) The incicrle of a non-isosceles $\triangle ABC$ is tangent to sies BC, CA, AB at A', B', C'. Lines AA', BB' intersect at P, AC and $A'C'$ at M, and lines $B'C'$ and BC at N. Prove that $IP \perp MN$.

These are very non-trivial problems; the last few are very hard.

8. (Iran TST 2007) The incircle ω of $\triangle ABC$ is tangent to AC, AB at E, F respectively. Points P, Q are on AB, AC such that PQ is parallel to BC and is tangent to ω. Prove that if M is the midpoint of PQ, and T the intersection point of EF and BC, then TM is tangent to ω.

9. (Romania TST 2007) The incircle ω of $\triangle ABC$ is tangent to AB, AC at F, E respectively. M is the midpoint of BC and N is the intersection of AM and EF. A circle Γ with diameter BC intersects BI, CI at X, Y respectively. Prove that $\frac{NX}{NY} = \frac{AC}{AB}$.

10. (Romania TST 2007) $\omega_a, \omega_b, \omega_c$ are circles inside $\triangle ABC$, that are tangent (externally) to each other, and ω_a is tangent to AB and AC, ω_b is tangent to BA and BC, and ω_c is tangent to CA and CB. Let D be the common point of ω_b and ω_c, E the common point of ω_c and ω_a, and F the common point of ω_a and ω_b. Show that AD, BE, CF are concurrent.

11. (Romania TST 2006) Let ABC be an acute triangle with $AB \neq AC$. Let D be the foot of the altitude from A and Γ the circumcircle of the triangle. Let ω_1 be the circle tangent to AD, BD and Γ. Let ω_2 be the circle tangent to AD, CD and Γ. Let l be the interior common tangent to both ω_1 and ω_2, different from AD. Prove that l passes through the midpoint of BC iff $AB + AC = 2BC$.

12. (China TST 2006 Generalized) In a cyclic quadrilateral $ABCD$ circumscribed about a circle with centre O, the diagonals AC, BD intersect at E. P is an arbitrary point inside $ABCD$ and X, Y, Z, W are the circumcentres of triangles ABP, BCP, CDP, DAP respectively. Show that XZ, YW, OE are concurrent.

13. (Iran TST 2009) The incircle of $\triangle ABC$ is tangent to BC, CA, AB at D, E, F respectively. Let M be the foot of the perpendicular from D to EF and P be the midpoint of DM. If H is the orthocenter of $\triangle BIC$, prove that PH bisects EF.

14. (SL 2007 G8) Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of $\triangle CPD$, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. The lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

15. (SL 2008 G7) Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of $\triangle ABC$ and $\triangle ADC$ by k_1 and k_2 respectively. Suppose that there exists a circle k tangent to lines AD, CD, to ray BA beyond A and to the ray BC beyond C. Prove that the common external tangents to k_1 and k_2 intersect on k.

16. (Iran TST 2010) Circles ω_1, ω_2 intersect at P, K. Points X, Y are on ω_1, ω_2 respectively so that XY is tangent externally to both circles and XY is closer to P than K. XP intersects ω_2 for the second time at C and YP intersects ω_1 for the second time at B. BX and CY intersect at A. Prove that if Q is the second intersection point of circumcircles of $\triangle ABC$ and $\triangle AXY$ then $\angle QXA = \angle QKP$.

Hints

1-3. Straightforward.

4. Extend KE to meet Γ at M. What can you say about A, I, M?

5. Use problem 4. What can you say about $\angle O_1DO_2$?

6. Lots of circles and points of tangency... Which theorem to use?

7. Poles and Polars are BACK.

8. Let ω be tangent to BC at D. Let S be the point of intersection of AD with ω. What can you say about the relation between T and AD with respect to ω?

9. Prove that X, Y lie on EF.

10. Draw the centres of the circles. Which theorem(s) should you be using here?

11. It is obvious which theorem to use here. What can you say about O_1, D, M, O_2?

12. Let M be the point of intersection of circumcircles of $\triangle BPC, \triangle APD$ and N the point of intersection of circumcircles of $\triangle BPA, \triangle CPD$. Consider the circumcentre of $\triangle PNM$.

13. Harmonic division.

14. Consider the circle tangent to AB, BC, DA. Find two more circles. Again lots of circles...

15. Let ω_1, ω_2 be tangent to AC at J, L. Prove that $AJ = CL$. Draw some excircles. Draw some lines parallel to AC.

16. This is a hard and beautiful problem. Let O be the intersection of AQ and XY. We want to use the radical axis theorem... Where is the third circle?

References

1 *Yufei Zhao, Lemmas in Euclidean Geometry,*
`http://web.mit.edu/yufeiz/www/geolemmas.pdf`

2 *Various MathLinks Forum Posts; in particular posts by Cosmin Pohoata and luisgeometria,*
`http://www.artofproblemsolving.com/Forum/index.php`