Number Theory Marathon

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Chapter 1

Problems

1. (IMO 1975) Let $f(n)$ denote the sum of the digits of n. Find $f(f(f(4444^{4444})))$.

2. Prove that if p and p^2+8 are prime, then p^3+8p+2 is prime.

3. Find all positive integers n such that for all odd integers a. If $a^2 \leq n$, then $a|n$.

4. Find all $n \in \mathbb{Z}^+$ such that $2^n + n|8^n + n$.

5. Find all nonnegative integers n such that there are integers a and b with the property:
$$n^2 = a + b \text{ and } n^3 = a^2 + b^2$$

6. Find all pairs of integers (n, m) satisfying $3n^2 + 3n + 7 = m^3$.

7. a, b, c, d are integers such that:
$$a < b \leq c < d,\ ad = bc \text{ and } \sqrt{d} - \sqrt{a} \leq 1$$
Show that a is a perfect square.

8. Let $a_1, a_2, \ldots, a_n$ be n integers satisfying $\sum_{k=1}^{n} a_k = 2009$. Find $\{a_1, a_2, ..., a_n\}$ and n that makes the product $\prod_{k=1}^{n} a_k$ obtain its maximum possible value.

9. Do there exist 10 distinct integers such that the sum of any 9 of them is a perfect square.

10. Let n be an integer and let $S = \{n, n+1, n+2, \ldots, n+38\}$. Show that there exists in S an element such that the sum of its digits is a multiple of 11. Is this still true if we substitute S by $\{n, n+1, n+2, \ldots, n+37\}$.

11. Let a_n be a sequence defined as: $a_1 = 2$ and $a_{n+1} = \lfloor \frac{3a_n}{2} \rfloor$ $(n \geq 1)$. Prove that there exists infinite numbers n for which a_n can be odd and for which it can be even.

12. Find all integers $n > 1$ such that any prime divisor of $n^6 - 1$ is also a prime divisor of $(n^3 - 1)(n^2 - 1)$.

13. Prove that any number consisting of 2^n identical digits has atleast n distinct prime factors.

14. Let p be a prime number which can be expressed as a sum of two integer squares, prove that there are no other integer squares, whose sum is p.

15. Find all pairs of positive integers (a, b) such that $\frac{a}{b} + \frac{21b}{25a}$ is a positive integer.

16. Let x, y, z be positive integers such that $(x, y, z) = 1$ and $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$. Prove that $x + y, x - z$ and $y - z$ are perfect squares.

17. Solve the following diophantine equation in natural numbers:
$$x^2 = 1 + y + y^2 + y^3 + y^4$$

18. Find all non negative integers m such that $(2^{2m+1})^2 + 1$ has at most two prime divisors.

19. Find all pairs of positive integers (a, b) such that $a - b$ is a prime and ab is a perfect square.

20. Let $n \geq 1$ be an integer. Suppose that we can find $2k$ distinct positive integers such that the sums $a_1 + b_1, \ldots, a_k + b_k$ are all distinct and strictly less than n. Prove that $k \leq \frac{2n-3}{5}$.

21. Let $\{F_i\}_{i=1}^{\infty}$ be the Fibonacci numbers ($F_1 = F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i > 2$). Prove that $F_1^2 + F_2^2 + ... + F_n^2 = \frac{1}{5}\left(F_{2n+3} - F_{2n-1} - (-1)^n\right)$.

22. If a, b, c are positive integers where $a = 20$ and $b|c$. Find all triples (a, b, c) such that a, b, c are in Harmonic Progression.

23. Find all triples (a, b, c) of natural numbers such that $lcm(a, b, c) = a + b + c$.

24. Find all natural numbers n such that n equals the cube of the sum of its digits.

25. Prove that:

    (a) If p is a prime divisor of the n^{th} Fermat number, then $p - 1$ is divisible by 2^{n+1}.

    (b) If a, n are positive integers where $a > 1$ and p is a prime divisor of $a^{2^n} + 1$, then $p - 1$ is divisible by 2^{n+1}.

26. Prove that $2^n + 1$ has no prime factors of the form $8k + 7$.

27. Prove that if for an integer $n > 1$ and a prime p we have $n|p - 1$ and $p|n^3 - 1$ then $4p - 3$ is a perfect square.

28. Find all odd integers n for which $n|3^n + 1$.

29. Prove that for any positive integer $n > 11$, $n^2 - 19n + 89$ is not a perfect square.

30. Suppose that $p > 3$ is a prime number. Prove that $7^p - 6^p - 1$ is divisible by 43.

31. Find the smallest positive integer a, such that $1971|50^n + a \cdot 23^n$ for any odd positive integer n.

32. Determine the greatest common divisor of the elements of the set $\{n^{13} - n | n \in \mathbb{Z}\}$

33. Prove that for any integer $n > 7$, $\binom{n}{7} - \lfloor \frac{n}{7} \rfloor$ is divisible by 7

34. Let a, b, c be non-zero integers, such that $a \neq c$ and $\frac{a}{c} = \frac{a^2+b^2}{c^2+b^2}$. Prove that $a^2 + b^2 + c^2$ cannot be prime.

35. For every positive integer n, prove that $n!$ is a divisor of $\prod_{k=0}^{n-1}(2^n - 2^k)$.

36. Show that the product of four consecutive positive integers cannot be a perfect square.

37. Prove the existence of infinitel many positive integers m for which $y = n^4 + m$ is not a prime for any positive integer n.

38. Show that for any positive integers $m, n > 2$: $2^m - 1 \nmid 2^n + 1$

39. For positive integer n, find all positive integers a such that $0 < a < 10^n$ and $10^n + a|10^{n+1} + a$ in terms of n.

40. Prove that there are infinitely many positive integers containing odd digits which are divisible by the sum of their digits.

41. If an integer n is such that $7n$ is of the form $a^2 + 3b^2$, prove that n is also of that form.

42. Prove that $p|1^k + 2^k + 3^k + \ldots + (n-1)^k$ if $p - 1 \nmid k$, p is a prime divisor of $n (n \geq 2)$, and k is a positive integer.

43. Find all non-negative solutions to: $43^n - 2^x 3^y 7^z = 1$

44. Find an unbounded increasing sequence of integers $\{a_i\}_{i=1}^{\infty}$ where for all positive integers i there exists positive integers a, b such that $2a_i|a+b$ and $ab = 4a_i(a_i+1)$

45. Prove that for every prime p, there exists an integer x, such that

$$x^8 \equiv 16 \pmod{p}$$

46. Find all positive integers n for which $n^2 + 89n + 2010$ is a perfect square.

47. Find all positive n such that:

    (a) $2^n - 1$ is divisible by 7.

    (b) $2^n + 1$ is divisible by 7.

48. Let $\sum_{i=1}^{87} 11^{10^i} - 4 \equiv A \pmod 8), 0 \leq A \leq 7$. Find all positive integers n for which $529|An^2 + An + 305$.

49. Find all prime numbers p such that $p^2 + 11$ has exactly 6 divisors.

50. Prove that for any positive integer $n > 1, n^5 + n^4 + 1$ is not a prime.

51. Prove that for positive integers n, m, $2^n - 1$ is divisible by $(2^m - 1)^2$ if and only if n is divisible by $m(2^m - 1)$.

52. Let positive integers g and l be given with $g|l$. Prove that the number of pairs of positive integers x, y satisfying $gcd(x, y) = g$ and $lcm(x, y) = l$ is 2^k, where k is the number of distinct prime factors of $\frac{l}{g}$.

53. Let $a \in \mathbb{Z}^+, a > 10$ and $b \in \mathbb{Z}^+$, such that the number of primes less than a is greater than b. Prove that there exists a set of a consecutive positive integers with exactly b prime numbers.

54. Let p be a prime and $a, b, c \in \mathbb{Z}^+$, such that $p = a+b+c-1$ and $p|a^3+b^3+c^3-1$. Prove that $\min{(a, b, c)} = 1$

55. Prove that no set of 2010 consecutive positive integers can be partitioned into two subsets, each having the same product of the elements.

56. Prove that the equation $3^k = m^2 + n^2 + 1$ has infinitely many solutions in $\mathbb{Z}^+$.

57. (a) If $m \cdot \phi(m) = n \cdot \phi(n)$ for positive integers m, n. Prove that $m = n$.

    (b) Show that this result does not hold if ϕ is replaced by σ.

58. Prove that for every $p \geq 13$, there exists an integer x such that $p|x^{2^{n+1}} - 2^{2^n}$ for $n \geq 2$.

59. Find all primes p, q such that $pq|2^p + 2^q$.

60. A sequence of positive integers is defined by $a_0 = 1$, $a_{n+1} = a_n^2 + 1$ for each $n \geq 0$. Find $gcd(a_{999}, a_{2004})$.

61. Prove that there are infinitely many positive integers n that can be written in the form $n = a^2 + b^2$ and $n = c^3 + d^3$ but not in $n = e^6 + f^6$.

62. Prove that there exists no positive integers x and y such that $x^2 + y + 2$ and $y^2 + 4x$ are perfect squares.

63. Find all primes p such that there exists $m, n \in \mathbb{Z}^+$ for which $\begin{cases} p = m^2 + n^2 \\ p|m^3 + n^3 - 4 \end{cases}$

64. Let that a, b, c be positive integers such that $a, b \neq c$. Prove that the exists infinitely many primes p such that $p|a^n + b^n + c^n$ for some $n \in \mathbb{Z}^+$.

65. Given that 9^{4000} has 3817 digits and has a leftmost digit 9 (base 10). How many of the number $9^0, 9^1, 9^2, \ldots, 9^{4000}$ have leftmost digit 9.

66.

67. Let $A = 6^n$ for real n. Find all natural numbers n such that $n^{A+2} + n^{A+1} + 1$ is a prime number.

68. Prove that there are infinitely many positive integers n such that $(1^4+0.25)(2^4+0.25)(3^4+0.25)\cdots(n^4+0.25)$ is the square of a rational number.

69. The sequence a_n is defined by $a_1 = 19, a_2 = 98$ and $a_{n+2} = a_n - \frac{2}{a_{n+1}}$. Show that there exists a positive integer m for which $a_m = 0$ and determine m.

70. Let $n \geq 2$ be an integer. Prove that if $n|3^n + 4^n$, then $7|n$.

71. Prove the identity: $\frac{[a,b,c]^2}{[a,b].[b,c].[c,a]} = \frac{(a,b,c)^2}{(a,b).(b,c).(c,a)}$ for $a, b, c \in \mathbb{Z}^+$.

    (Note: (a, b) means gcd and $[a, b]$ means lcm)

72. Find all non-negative integers n such that $2^{200} + 2^{192}.15 + 2^n$ is a perfect square.

73. Find the smallest positive integer ending in 1986 which is divisible by 1987.

74. Find all positive integers n such that $n|3^n - 2^n$.

75.

76. Find all positive integers n such that $(n! + 1)! = 121(n^3 - n)!$.

77. (ISL 2005) Prove that if for all $n \in \mathbb{Z}^+$ we have $a^n + n|b^n + n$, then $a = b$.

78. Solve in $\mathbb{Z}$: $x^{2010} + 2010! = 251^y$

79. p, q, r are prime numbers such that $p|qr-1, q|pr-1, r|pq-1$. Find all possible values of pqr.

80. Find all $a (a \in \mathbb{Z}^+)$ for which $a^4 + 4^a$ is a prime number.

81. Find all pairs of positive integers (m, n) such that $2^m + 3^n$ is a perfect square.

82. Prove that for all $p \geq 3$ there is a positive integer such that:

$$\frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_p} = 1$$

and $n_i \neq n_j$ if $i \neq j$.

83. Find every positive integer k such that: $gcd(37m-1, 1998) = gcd(37m-1, k)$ for every positive integer m.

84. Given that $a_0 = 1, a_1 = 2$ and $i(i+1)a_{i+1} = i(i-1)a_i - (i-2)a_{i-1}$ for every integer $i \geq 1$. Find

$$\frac{a_0}{a_1} + \frac{a_1}{a_2} + \ldots + \frac{a_n}{a_{n+1}} \text{ for } n \in \mathbb{Z}^+$$

85. Find all positive integers a, b, c such that $gcd(a, b) = gcd(b, c) = gcd(c, a) = 1$, and $(a+b+c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) \in \mathbb{Z}^+$

86. Prove that for a prime p, every odd prime divisor of $3^p - 1$ is greater than p.

87. Prove that for any pair of positive integers a, n where a is odd: $2^{n+2}|a^{2^n} - 1$

88. Let a, b, n be positive integers such that for any $k \in \mathbb{Z}^+ - \{b\}$, $b - k|a - k^n$. Prove that $a = b^n$.

89. Find the least positive integer n such that the average of $1^2, 2^2, \ldots, n^2$ is a perfect square.

90. If $p = 2^n + 1$ for $n \geq 2$ is a prime, then prove that $p|3^{\frac{p-1}{2}} + 1$.

91. Prove that if $p = 2^{2^k} + 1$ for $k \geq 1$ and $p|3^{\frac{p-1}{2}} + 1$ then p is a prime.

92. Show that 2 is a primitive root modulo 3^n for any $n \in \mathbb{Z}^+$.

93. Find all $n \in \mathbb{Z}^+$ such that $n^2|(n-2)!$.

94. If p is a prime, and $\{a_1, a_2, \cdots a_p\}, \{b_1, b_2, \cdots b_p\}$ are complete sets of residues modulo p such that $\{a_1b_1, a_2b_2, \cdots a_pb_p\}$ is also a complete set of residues modulo p, then find all such p.

95. Find all positive integers n such that $n \cdot 2^{n-1} + 1$ is a perfect square.

96. Find all two-digit integers n for which the sum of the digits of $10^n - n$ is divisible by 170.

97. Prove that for every $n \in \mathbb{Z}^+$, there exist n consecutive integers such that each one is divisible by the square of an integer greater than 1.

98. Prove that if $\lfloor \frac{2^n}{n} \rfloor$ is a power of 2 for some positive integer n then n is a power of 2.

99. The integer n is positive. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying:
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$
Prove that n is a perfect square.

100. Find all the solutions (p, q, n), with p, q primes such that $p^2 + q^2 = pqn + 1$.

101. (Brazil 2005) Let b be an integer and let a, c be positive integers. Prove that there exists a positive integer x such that
$$a^x + x \equiv b \pmod{c}$$

102. Let x, y be two positive integers such that $3x^2 + x = 4y^2 + y$. Prove that $x - y$ is a perfect square.

103. Find all $n \in \mathbb{Z}^+$ such that $2^n + 3^n + 6^n = x^2$.

104. Find all $x, y \in \mathbb{Z}^+$ such that $3^x 7^y + 1$ is a perfect odd power.

105. Prove the following properties of $n!$:

    (a) If p is prime then $p^p | n! \iff p^{p+1} | n!$
    (b) There are no integers $a, b > 1$ such that $a^b = n!$

106. Let a, b, c be integers such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3$. Prove that abc is a perfect cube.

107. Let $x, y > 1$ be two positive integers such that $2x^2 - 1 = y^{15}$. Prove that $5|x$.

108. (France 1990)

    (a) Find all positive integer solutions of $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{4}$
    (b) Find all $n \in \mathbb{Z}^+$ such that there exists $x_1, x_2, ..., x_n \in \mathbb{Z}^+$ for which
$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + ... + \frac{1}{x_n^2} = 1$$

109. Find all integer solutions for $x^3 + 3 = 4y(y+1)$.

110. Do there exist $x, y, z \in \mathbb{Z}^+$ such that $2548^x + (-2005)^y = (-543)^z$?

111. Find all integer solutions of $3a^2 - 4b^3 = 7^c$ for $a, b, c \in \mathbb{Z}$ and $c \geq 0$.

112. (Iran 2010) Let $a, b \in \mathbb{Z}^+$ and $a > b$. We know that $gcd(a-b, ab+1) = 1$ and $gcd(a+b, ab-1) = 1$. Prove that $(a-b)^2 + (ab+1)^2$ is not a perfect square.

113. Find all integers a, b such that $7a + 14b = 5a^2 + 5ab + 5b^2$.

114. Find all pairs of positive integers x, y such that $\frac{xy^2}{x+y}$ is prime.

115. Let x, y be irrational numbers such that $xy, x^2 + y, y^2 + x$ are all rational.

    (a) Prove that there exist such x, y. (Give an example)

    (b) Find all possible values of $x + y$.

116. Find all primes p such that $1 + p \cdot 2^p$ is a perfect square.

117. How many divisors of 30^{2003} are not divisors of 20^{2000}?

118. Let n be a positive integer and let k be an odd positive integer. Moreover, let a, b, c be integers such that $a^n + kb = b^n + kc = c^n + ka$. Prove that $a = b = c$.

119. Prove that there are infinitely many positive integers n such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$.

120. Prove that the equation $x^2y^2 = z^2(z^2 - x^2 - y^2)$ has no solution in positive integers.

121. Solve in $\mathbb{Z}^+$: $n! - 3n + 28 = k^2$

122. (Balkan MO 1999) Given a prime $p \equiv 2 \pmod 3$, and the equation $y^2 - x^3 - 1 \equiv 0 \pmod p$ with $1 \leq x, y \leq p-1$. Prove that the number of solutions (x, y) of this congruence is at most $p - 1$.

123. For two positive integers a, b which are relatively prime, find all integers that can be the great common divisor of $a + b$ and $\frac{a^{2005}+b^{2005}}{a+b}$.

124. If $p \equiv 1 \pmod 4$ is a prime and if $a, b \in \mathbb{Z}^+$ such that $a^2 + b^2 = p$, where a is odd. Then prove that there exists $x \in \mathbb{Z}$ such that $x^2 \equiv a \pmod p$.

125. Prove that the numbers $\binom{2^n-1}{0}, \binom{2^n-1}{1}, \cdots, \binom{2^n-1}{2^{n-1}-1}$ form a reduced residue system modulo 2^n.

126. If $x^{2p} + y^{2p} + z^{2p} = t^{2p}$ where x, y, z, t are integers $p \equiv 3 \pmod 4$ is a prime. Then prove that at least one of the numbers x, y, z, t is divisible by p.

127. I write the number 6 on the blackboard. At the n^{th} step, an integer k on the board is replaced by $k + (n, k)$. Prove that at each step, the number on the blackboard increases either by 1 or by a prime number.

128. Prove that in Fibonacci sequence $4|\varphi(F_n)$ for any $n > 4$.

129. Find all primes a, b, c such that $ab + bc + ca > abc$.

130. Show that there exists $n > 2$ such that $\underbrace{199 \cdots 991}_{n\ nines}$ is divisible by 1991.

131. $a, b, c, d \in R^+$ satisfy $\lfloor na \rfloor + \lfloor nb \rfloor = \lfloor nc \rfloor + \lfloor nd \rfloor$ for every $n \in \mathbb{Z}^+$. Prove that $a + b = c + d$.

132. Show that $ax^2 + bx + c = 0$ has no rational solutions if a, b, c are odd integers.

133. Find all irrational positive numbers $a > 1$ such that for a given integer $n > 2$ we have $\sqrt[n]{a + \sqrt{a^2 - 1}} + \sqrt[n]{a - \sqrt{a^2 - 1}}$ to be rational.

134. $a, b, c \in \mathbb{Z}$ are given. Prove that these statements are equivalent:

   (a) There are infinitely many primes q satisfying $\left(\frac{a}{q}\right) = \left(\frac{b}{q}\right) = \left(\frac{c}{q}\right) = -1$.

   (b) a, b, c and abc are not perfect squares.

135. $x = a^2$ where $a \in \mathbb{Z}^+$ and $y|2a$ where $y > 0$. Prove that $x + y$ can not be a perfect square.

136. The sequence (a_n) is defined such that $a_0 = 2, a_{n+1} = 4a_n + \sqrt{15{a_n}^2 - 60}$.

   (a) Find the general term a_n.

   (b) Prove that $\frac{1}{5}.(a_{2n} + 8)$ can be expressed in the form of sum of squares of three consecutive integers for all integers $n \geq 1$.

137. Solve in $\mathbb{Z}$: $a^3 + b^3 = 9$

138. Prove that if $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ is a perfet number, then

$$2 < \prod_{i=1}^{n} \frac{p_i}{p_i - 1} < 4$$

Moreover, if a is odd the upper bound may be reduced to $2\sqrt[3]{2}$.

139. Prove that for all $n \in \mathbb{Z}^+$, the following inequality holds:

$$\phi(n)\tau(n) \geq n$$

140. Let p be a prime, prove that: $\left((\frac{p-1}{2})!\right)^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p}$

141. Find all integers k, not divisible by 4, such that $3^k - 4$ and $k+1$ are both prime.

142. Given n and $\{a_i\}_{i=1}^n$, such that $a_1 + a_2 + ...a_n$. Is it possible for the product $a_1 a_2 ... a_n$ to have a maximum value? If so, what is that maximum value?

143. Let $a, b, c \in \mathbb{Z}^+$, such that $a + b + c = 2pq(p^{30} - q^{30})$ for some positive integers p and q.

    (a) Prove that $k = a^3 + b^3 + c^3$ is not prime.

    (b) Prove that if abc is maximum, then $1984|k$.

144. Solve for $x, y, z \in \mathbb{Z}$:

$$\begin{cases} x + y & = & 1 - z \\ x^3 + y^3 & = & 1 - z^3 \end{cases}$$

145. Determine the greatest common divisor of the numbers in the set:

$$\{16^n + 10n - 1 | n \in \mathbb{Z}^+\}$$

146. For a three-digit number $\overline{abc}$ it is known that $k = \frac{\overline{abc}}{a^3+b^3+c^3}$ is an integer. Find all possible values of k.

147. If a, b, c are digits. Prove that if $(a + b)(b + c)(c + a) \mid \overline{abc}$, then $abc = 0$.

148. Let m, n be two positive integers. Prove that if there exists infinitely many integers k such that $k^2 + 2kn + m^2$ is a square, then $m = n$.

149. Prove that if $p_1, p_2, \ldots, p_n$ are primes and $p_i \neq 2$ $(1 \leq i \leq n)$, then

$$6 \Big| \sum_{i=1}^{n} p_i^2 \implies 6|n$$

150. Prove that there are infinitely many primes that leave remainder 1 when divided by 4, without using Dirichlet's theorem.

151. Prove that there are infinitely many primes that leave remainder 3 when divided by 4, without using Dirichlet's theorem.

152. Prove that there are infinitely many primes that leave remainder -1 when divided by 13, without using Dirichlet's theorem.

153. Prove that there are infinitely many primes that leave remainder -1 when divided by 13, without using Dirichlet's theorem.

154. Solve in $\mathbb{Z}^+$: $n_1^5 + n_2^5 + ... + n_9^5 = 161061$

155. Anne has n marbles, when she arranges them in rows of 6 there's 3 left over, and when she arranges them in rows of 9, there's 6 left over. What is the smallest possible value of n?

156. Find all nonnegative integer solutions for $x^2 + y^2 + z^2 = 2xyz$.

157. If $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, where a, b, c are positive integers with no common factor, prove that $a + b$ is a square.

158. Suppose that n can be written as a sum of two squares of positive integers in two different ways. Show that the number n is composite.

159. (Hungary 2000) Find all quatuples of positive integers (a, b, p, n) such that $a^3 + b^3 = p^n$ where p is a prime.

160. (Argentina TST 2009) Find all positive integers n such that $20^n - 13^n - 7^n$ is divisible by 309.

161. Show that for any $n \in \mathbb{Z}^+$: $2^n \nmid n!$

162. All the subsets of $\{1, 2, \ldots, 2010\}$ that do not contain two consecutive members are formed. The product of the numbers in each subset is calculated. Prove that the sum of squares of these products is equal to $2011! - 1$.

163. Prove that the equation $x^2 + (x + 1)^2 = y^3$ does not have any solution in $\mathbb{Z}^+$.

164. Find the smallest positive prime that divides $n^2 + 5n + 23$ for some integer n.

165. Find all triples (a, b, c) of distinct positive integers such that $a + b + c | 3abc$.

166. Find the least positive integer m for which: $\overbrace{100^{100^{100^{\cdot^{\cdot^{\cdot^{100}}}}}}}^{m} > \overbrace{3^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}^{100}$

167. Prove that for any positive integer n, there exist a prime p and positive integer m such that:

    (a) $p \equiv 5 \pmod 6$
    (b) $p \nmid n$
    (c) $n \equiv m^3 \pmod p$

168. Prove that, if n is the product of different prime numbers of the form $2^k - 1$, $k \in \mathbb{Z}^+$, then $\sigma(n)$ is a perfect power of 2.

169. $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n; n \geq 1$. For $n \geq 2$ and $x \in \mathbb{R}$. Prove that

$$\sum_{k=1}^{n} |x-k|F_k \geq F_{n+2} + F_n - n - 1$$

170. Find all pairs $x, y \in \mathbb{Z}$ such that $\frac{y^3-y}{x^3-x} = 2$.

171. Prove that there are infinitely many pairs of positive integers x and y such that $\frac{x^3+y^3+2}{x^2+y^2+1}$ is twice a perfect square.

172. (APMO 1998) Find the largest integer n which is divisible by all positive integers less $\sqrt[3]{n}$.

173. Let p and q be different primes. Prove that: $\left\lfloor \frac{p^q+q^p}{pq} \right\rfloor$ is even if $p, q \neq 2$.

174. For relative prime positive integers m, n. Prove that

$$\frac{\log_a m}{\log_a n}$$

is not rational for any $a \in \mathbb{Z}^+$.

175. Let $\{a_i\}_{i=1}^{\infty}$ be a strictly increasing sequence satisfying: $\begin{cases} \forall n \geq 1 : a_n \in \mathbb{Z}^+ \\ a_{2n} = a_n + n \end{cases}$

Suppose that if a_n is prime then n is prime too. Find a_{1993}.

176. Prove that if p is a prime, then $p^p - 1$ has a prime factor that is congruent to 1 modulo p.

177. Find all $x, y, z \in \mathbb{Z}^+$ such that $x^2 = y^z - 3$ and $z \not\equiv 1 \pmod 4$.

178. Prove that there are infinitely many sets of five consecutive positive integers a, b, c, d, e such that $a+b+c+d+e$ is a perfect cube and $b+c+d$ is a perfect square.

179. Show that $N = 101010\ldots101$ is not a prime, except when $N = 101$.

180. Solve the equation $3^a = 2^a b + 1$ where $a, b \in \mathbb{Z}^+$.

181. Let k be a positive integer such that $k > 1$. Consider the sequence $\{a_n\}_{n=1}^{\infty} = 2^{2^n} + k$. prove that there are infinitely many positive integers n such that a_n is a composite number.

182. Let x be an integer, and p, q prime numbers greater than two such that $q \nmid x-1$ and $q|x^p - 1$. Prove that $q \mid (1+x)...(1+x^{p-1}) - 1$.

183. Prove that $a^m \equiv a^{m-\phi(m)} \pmod{m}$

184. Find all positive integers x, y such that $x(y+2)$ is the product of two primes and $x^2 + 3x + 3 = (11 - x + y)^3$.

185. Solve in $\mathbb{Z}$: $2^x + 1 = 3^y$

186. Let $a, b, c \in \mathbb{Z}^+$ such that $a^2 + b^2 = c^2$. Prove that $60|abc$.

187. Let a, b, c, d, n are positive integers such that $ab = cd$. Prove that $a^n + b^n + c^n + d^n$ can't be prime number.

188. Solve in $\mathbb{Z}^+$: $a! + 1 = (a+1)^b$

189. Let a, b be two positive integers such that $ab(a+b)$ is divisible by $a^2 + ab + b^2$. Prove that

$$|a - b| > \sqrt[3]{ab}$$

190. Let σ be the sum of divisors function, and a, b positive integers, such that $\sigma(a) = \sigma(b) = a + b$

   (a) Find the two smallest values of $a + b$.

   (b) Find the general solutions (a, b).

191. Let n be a non-negative integer. Find the non-negative integers a, b, c, d such that:

$$a^2 + b^2 + c^2 + d^2 = 7 \cdot 4^n$$

192. If p is a prime number, show that the coefficients of the terms of $(1+x)^{p-1}$ are alternately greater and lesser than a multiple of p by 1.

193. If p and q are any two positive integers, show that $(pq)!$ is divisible by $(q!)^p \cdot p!$.

194. Let n and k be given integers, $n > 0$ and $k(n-1)$ is even. Prove that there exists $x, y \in \mathbb{Z}^+$ such that $gcd(x, n) = gcd(y, n) = 1$ and $x + y \equiv k \pmod{n}$.

195. (IMO SL 1992) Prove that $\frac{5^{125}-1}{5^{25}-1}$ is composite.

196. Let k be a positive integer and let $p = 4k + 1$ be a prime number. Prove that the number $\frac{p^{p^3}-1}{p^{p^2}-1}$ is composite.

197. a, b, c are positive integers such that $a - b$ is prime and $3c^2 = c(a+b) + ab$. Prove that $8c + 1$ is a perfect square.

198. Find the smallest prime divisor of $12^{2^{15}} + 1$

199. For every positive integer n, prove that:

$$\frac{\sigma(1)}{1} + \frac{\sigma(2)}{2} + ... + \frac{\sigma(n)}{n} \leq 2n$$

200. (UK 1998) Let x, y, z be positive integer such that $\frac{1}{x} - \frac{1}{y} = \frac{1}{z}$ Prove that $gcd(x, y, z) \cdot xyz$ and $gcd(x, y, z) \cdot (y - x)$ are perfect number

201. Positive integers n and $a > 1$ are given. Prove that the equation $a^x \equiv x \pmod{n}$ has got infinitely many solutions in $\mathbb{Z}^+$.

202. Prove that the polynomial

$$f(x) = \frac{x^n + x^m - 2}{x^{gcd(n,m)} - 1}$$

is irreducible over $\mathbb{Q}$ for all integers $n > m > 0$.

Chapter 2

Solutions

1. $$4444^{4444} < 10000^{4444} = 10^{17776}$$

   So 4444^{4444} has at most 17776 digits, which means that $f(N)$ can not be greater than $9 \cdot 17776 = 159984$ Here we see that $f(f(N))$ can not be greater than $9 \cdot 5 = 45$, $f(f(f(N)))$ can not be greater than $3 + 9 = 12$.

   Also, we know that

   $$f(f(f(N))) \equiv f(f(N)) \equiv f(N) \equiv N \pmod 9$$

   And

   $$4444^{4444} \equiv (-2)^{4444} = 2^{4444} = 2^{4440} \cdot 2^4 = 64^{740} \cdot 16 \equiv 1 \cdot 7 \equiv 7 \pmod 9$$

   Since $f(f(f(N))) \leq 12$, we must have $\boxed{f(f(f(N))) = 7}$

2. If $p = 2$ and $p = 3$ we can check that the result is valid.

   If $p > 3$ then we have $p \not\equiv 0 \pmod 3$, and $p^2 \equiv 1 \pmod 3$, which means $p^2 + 8 \equiv 0 \pmod 3$, so $p^2 + 8$ can't be prime.

3. We'll consider some cases:

   (a) $1^2 \leq n < 3^2$: $1|n$, $\boxed{n = 1, 2, 3, 4, 5, 6, 7, 8}$

   (b) $3^2 \leq n < 5^2$: $lcm(1, 3) = 3|n$, $\boxed{n = 9, 12, 15, 18, 21, 24}$

   (c) $5^2 \leq n < 7^2$: $lcm(1, 3, 5) = 15|n$, $\boxed{n = 30, 45}$

   Lemma 1. *If $k \geq 4$:*

   $$lcm(1, 3, \ldots, 2k - 3, 2k - 1) > (2k + 1)^2$$

Proof.

Since $2k-1$, $2k-3$, $2k-5$ are coprime.

$lcm(1,3,\ldots,2k-3,2k-1) \geq lcm(2k-1,2k-3,2k-5) = (2k-1)(2k-3)(2k-5)$

$lcm(1,3,\ldots,2k-3,2k-1) \geq 7(2k-3)(2k-5) > 6(2k-3)(2k-5)$

$lcm(1,3,\ldots,2k-3,2k-1) = (4k-6)(6k-5) > (2k+1)(2k+1) = (2k+1)^2$

$$\nabla$$

So, there isn't a solution for $n \geq 7^2$.

4. $2^n + n | 8^n + n$

   $\Leftrightarrow\ 2^n + n | (4^n)(2^n + n) - (8^n + n) = n \cdot 4^n - n$

   $\Leftrightarrow\ 2^n + n | (n \cdot 2^n)(2^n + n) - (n \cdot 4^n - n) = n^2 \cdot 2^n + n$

   $\Leftrightarrow\ 2^n + n | (n^2)(2^n + n) - (n^2 \cdot 2^n + n) = n^3 - n$

   For $1 \leq n \leq 9$, we get solutions $\boxed{n = 1, 2, 4, 6}$.

   For $n \geq 10$, we'll prove by induction that $n^3 - n < 2^n + n$, so there won't be any other solution:

   (a) $n = 10$: $1000 - 10 = 900 < 1024 + 10 = 1034$
   (b) Assume it is true for $n = k$, so $k^3 - 2k < 2^k$
   (c) $n = k + 1$:

$$2^k > k^3 - 2k \geq 10k^2 - 2k \geq 3k^2 + 68k > 3k^2 + 3k - 1$$

$$2^k + 2^k > (k^3 - 2k) + (3k^2 + 3k - 1) = (k+1)^3 - 2(k+1)$$

$$2^{k+1} + (k+1) > (k+1)^3 - (k+1)$$

5. $(a-b)^2 \geq 0\ \Rightarrow 2(a^2 + b^2) \geq (a+b)^2\ \Rightarrow 2n^3 \geq n^4\ \Rightarrow n^3(2-n) \geq 0$

   Since $n \geq 0$, $2 - n \geq 0\ \Rightarrow n \leq 2$

   $0^2 = 0 + 0$, $0^3 = 0^2 + 0^2$

   $1^2 = 1 + 0$, $1^3 = 1^2 + 0^2$

   $2^2 = 2 + 2$, $2^3 = 2^2 + 2^2$

   So, $\boxed{n = 0, 1, 2}$

6. $3n^2 + 3n + 7 \equiv 1 \pmod 3 \Rightarrow m^3 \equiv 1 \pmod 3 \Rightarrow m \equiv 1 \pmod 3$

   So, $m = 3k + 1$, where k is a nonnegative integer.

   Replacing in the equation:

$$3n^2 + 3n + 7 = (3k + 1)^3$$

$$n^2 + n + 2 = 9k^3 + 9k^2 + 3k$$

$$n^2 + n + 2 \equiv 0 \pmod 3$$

   We've got to consider three cases:

   (a) $n \equiv 0 \pmod 3$: $n^2 + n + 2 \equiv 2 \pmod 3$
   (b) $n \equiv 1 \pmod 3$: $n^2 + n + 2 \equiv 1 \pmod 3$
   (c) $n \equiv 2 \pmod 3$: $n^2 + n + 2 \equiv 2 \pmod 3$

   So, there is no solution to the equation.

7. We begin stating the Factoring Lemma:

   Lemma 2. *Let a, b, c, d be positive integers with $ab = cd$. Then there are positive integers m, n, p, q such that $gcd(n, p) = 1$ and*

$$a = mn,\ b = pq,\ c = mp,\ d = nq$$

   Proof.

   The condition $ab = cd$ can be rewritten as $\frac{a}{c} = \frac{d}{b}$, both fractions have the same representation $\frac{n}{p}$ in lowest terms.

$$m = \frac{a}{n} = \frac{c}{p},\ q = \frac{d}{n} = \frac{b}{p}$$

$$a = mn,\ b = pq,\ c = mp,\ d = nq$$

$$\nabla$$

   So: $a = mn$, $b = pq$, $c = mp$, $d = nq$, and $mn < pq \leq mp < nq$, which implies $q > m$ and $n > p$.

$$d = nq \geq (m + 1)(n + 1)$$

$$d \geq mn + m + n + 1$$

$$d \geq mn + 2\sqrt{mn} + 1 \text{ (AM-GM)}$$
$$d \geq (\sqrt{mn} + 1)^2 = (\sqrt{a} + 1)^2$$
$$\sqrt{d} - \sqrt{a} \geq 1$$

But we also have $\sqrt{d} - \sqrt{a} \leq 1$, so from the equality case of AM-GM, we must have $m = n$, hence a is a square.

8. We will prove that in order to reach the maximum, all a_i are either 2 or 3.

   (a) If $a_i = 1$: we can take $a_j (j \neq i)$ and replace both by $a_j + 1$ and this will increase the product.

   (b) If $a_i \geq 4$: we can replace it by 2 and $a_i - 2$, $2(a_i - 2) \geq a_i$, so this will increase the product.

   Also there can be at most two 2's since if there were more we could take three 2's and replace them by two 3's, because $2 + 2 + 2 = 3 + 3$ and $2^3 < 3^2$.

   Hence the maximum product has the form $2^a \cdot 3^b$ where a, b are nonnegative integers such that $a < 3$. The only solution is $a = 1, b = 669$. So the maximum product is $2 \cdot 3^{669}$

   Comment. 2009 can be replace for any positive integer $N \geq 2$ and the solution to our last equation will always be unique since $N - 0 \cdot 2$, $N - 1 \cdot 2$ and $N - 2 \cdot 3$ are different modulo 3.

9. Let $m_1, m_2, \ldots, m_{10}$ be different integer numbers. Then

$$a_i = (\sum_{j=1}^{10} 9m_j^2) - 81{m_i}^2$$

   is a solution to the problem.

10. In this set there are at least 3 multiple of 10, let these be $\lambda_1, \lambda_2, \lambda_3$, such that $n \leq \lambda_1 \leq n + 9$, $\lambda_2 = \lambda_1 + 10$ and $\lambda_3 = \lambda_2 + 10$.

    Let $S(n)$ denote the sum of the digits of n.

    (a) If $S(\lambda_i) \equiv 0 \pmod{11}$ we are done.

    (b) If $S(\lambda_i) \not\equiv 1 \pmod{11}$ with $i \in \{1, 2\}$ then $S(\lambda_1 + 11 - k) \equiv 0 \pmod{11}$.

    (c) If $S(\lambda_1), S(\lambda_2) \equiv 1 \pmod{11}$, let d be the second last digit of λ_2:

      i. If $d \geq 1$ then $S(\lambda_1) \equiv 0 \pmod{11}$

      ii. If $d = 0$ then $S(\lambda_2 + 19) \equiv 1 + 1 + 9 \equiv 0 \pmod{11}$

    Concerning the set $\{n, n+1, n+2, \ldots, n+37\}$ we can find the following counterexample $\{999981, \ldots, 1000018\}$ where there are not such a number which sum of its digits is divisible by 11.